Introduction to the Lattice Boltzmann Method

Bastien Chopard

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Concepts

- Inspired from the LGA approach
- Directly simulate on the computer the average value $N_i = < n_i >$.
- There is no need for an underlying Boolean model of particles
- Much more flexibility to choose the collision term $\Omega_i(N)$

The Lattice Boltzmann (LB) dynamics

From now on, we use the notation f_i instead of N_i.

The LB scheme has a collision step

$$f_i^{out} = f_i(\vec{r}, t) + \Omega_i(f(\vec{r}, t))$$

and a propagation step

$$f_i(\vec{r} + \Delta t \vec{v}_i, t + 1) = f_i^{out}$$

where Ω must obey mass and momentum conservation

$$\sum_{i=0}^{z} \Omega_i = 0 \qquad \sum_{i=0}^{z} \Omega_i \vec{v}_i = 0$$

Macroscopic quantities

As before, the fluid density ρ and the fluid speed $\vec{u}$ are defined as

$$\rho = \sum_{i=0}^{z} f_i \qquad \rho\vec{u} = \sum_{i=0}^{z} f_i \vec{v}_i$$

The equation governing the f_i's is

$$f_i(\vec{r} + \Delta t \vec{v}_i, t + 1) - f_i(\vec{r}, t) = \Omega_i(f)$$

With an explicit expression for Ω, the multiscale Chapman-Enskog expansion can be repeated as for LGA models and the continuity and Navier-Stokes equation can be calculated.

Single relaxation time model (BGK)

We choose

$$\Omega_i = \omega \left(f_i^{eq}(\rho, \vec{u}) - f_i \right)$$

where $\omega = \frac{1}{\tau}$ is a free parameter (inverse of a relaxation time) and $f_i^{eq}(\rho, \vec{u})$ is taken as

$$f_i^{eq} = \rho t_i \left[1 + \frac{\vec{v}_i \vec{u}}{\bar{c}_s^2} + \frac{1}{2\bar{c}_s^4} Q_{i\alpha\beta} u_\alpha u_b \right]$$

where $\bar{c}_s$ and t_i are lattice specific constant and

$$Q_{i\alpha\beta} = v_{i\alpha} v_{i\beta} - \bar{c}_s^2 \delta_{\alpha\beta}$$

Momentum Tensor

The local equilibrium coefficients are such that

$$\Pi_{\alpha\beta}^{eq} = \sum_{i=0}^{z} f_i^{eq} v_{i\alpha} v_{i\beta} = \rho c_s^2 \delta_{\alpha\beta} + \rho u_\alpha u_\beta$$

because, by definition, the t_i's obey

$$\sum_i t_i v_{i\alpha} v_{i\beta} = \bar{c}_s^2 \delta_{\alpha\beta}$$

and

$$\sum_i t_i v_{i\alpha} v_{i\beta} v_{i\gamma} v_{i\delta} = \bar{c}_s^4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}$$

Table of coefficients

topology	t_0	t_i (slow speeds)	t_i (fast speeds)	$\bar{c}_s^2/v^2$
D2Q9	$4/9$	$1/9$	$1/36$	$1/3$
D3Q19	$1/3$	$1/18$	$1/36$	$1/3$

Equilibrium part of f_i

The Chapman-Enskog expansion is

$$f = f^0 + f^{(1)}$$

The condition $\Omega(f^{(0)}) = 0$ (see LGA slides) implies that

$$f^{(0)} = f^{eq}$$

and thus

$$\rho = \sum_i f_i^{eq} \qquad \rho\vec{u} = \sum_i f_i^{eq}\vec{v}_i$$

which can be verified from the expression of f^{eq}.

Non-Equilibrium part of f_i

We have

$$\Omega(f^{eq} + f^{(1)}) = \omega \left[f^{eq} - (f^{eq} + f^{(1)}) \right] = -\omega f^{(1)}$$

Thus the first order Chapman-Enskog equation gives

$$-\frac{\omega}{\Delta t} f^{(1)} = \partial_t f_i^{(0)} + \vec{v}_i \cdot \mathrm{grad} f_i^{(0)} + O(\Delta t)$$

where we use

$$\partial_t f_i^{(0)} = \frac{\partial f_i^{(0)}}{\partial \rho} \partial_t \rho + \frac{\partial f_i^{(0)}}{\partial \rho u_\alpha} \partial_t \rho u_\alpha$$

with

$$\partial_t \rho = -\mathrm{div} \rho \vec{u} \qquad \partial_t \rho u_\alpha = -\partial_\beta \Pi_{\alpha\beta}^{(0)}$$

Non-equilibrium part

Using the expression for f^{eq}, we obtain

$$f_i^{neq} \approx f_i^{(1)} = -\frac{\Delta t}{\omega c_s^2} t_i Q_{i\alpha\beta} \partial_a \rho u_\beta$$

and

$$\Pi_{\alpha\beta}^{neq} \approx \sum_{i=0}^{z} f_i^{neq} v_{i\alpha} v_{i\beta} = -\frac{\Delta t}{\omega} c_s^2 \left(\partial_a \rho u_\beta + \partial_b \rho u_\alpha\right)$$

Full Navier-Stokes equation (1)

Using second order in Δt and multiscale Chapman-Enskog expansion yield the following equation for momentum conservation

$$\partial_t \rho u_\alpha + \partial_\beta \left[\Pi^{(0)}_{\alpha\beta} + \Pi^{(1)}_{\alpha\beta} + \frac{\Delta t}{2} \left(\partial_t \Pi^{(0)}_{\alpha\beta} + \partial_\gamma S^{(0)}_{\alpha\beta\gamma} \right) \right] = 0$$

where $\partial_t \Pi^{(0)}_{\alpha\beta}$ and

$$S^{(0)}_{\alpha\beta\gamma} = \sum_{i=1}^{z} v_{i\alpha} v_{i\beta} v_{i\gamma} f^{(0)}_i$$

accounts for the **lattice contributions** to the momentum transfert.

Full Navier-Stokes equation (2)

The final calculation gives

$$\partial_t \rho u_\alpha + \rho u_\beta \partial_\beta u_\alpha \quad + \quad u_\alpha \mathrm{div} \rho \vec{u} = -c_s^2 \partial_\alpha \rho + \Delta t \bar{c}_s^2 \left(\frac{1}{\omega} - \frac{1}{2} \right) \nabla^2 \rho u_\alpha + \Delta t \left(\frac{1}{\omega} - \frac{1}{2} \right) \left[2\bar{c}_s^2 - c_s^2 \right] \partial_\alpha \mathrm{div} \rho \vec{u} \qquad (1)$$

Incompressible limit

At low Mach number, we can assume that $\mathrm{div}\rho\vec{u} = 0$ and one recovers the usual Navier-Stokes equation

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu_{\mathrm{lb}} \nabla^2 \vec{u} \tag{2}$$

where $p = c_s^2 \rho$ is the scalar pressure and ν_{lb} is the kinematic viscosity

$$\nu_{\mathrm{lb}} = \Delta t \bar{c}_s^2 \left(\frac{1}{\omega} - \frac{1}{2} \right) \tag{3}$$

Body force

A **constant body force** can be included as

$$f_i(\vec{r} + \Delta t \vec{v}_i, t + \Delta t) = \omega f_i^{(0)}(\vec{r}, t) + (1 - \omega)\, f_i(\vec{r}, t) + \frac{\Delta t}{\bar{c}_s^2} \vec{v}_i \cdot \vec{F} \quad (4)$$

so as to produce the following correction to Navier-Stokes

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho} \nabla p + \nu_{\mathrm{lb}} \nabla^2 \vec{u} + \vec{F}$$

Boundary Conditions

Several boundary conditions can be used on a wall (with imposed velocity)

- Bounce back (no-slip): not very accurate
- Zhou and He: simple and usually efficient: need to compute ρ on the wall and compute the missing distribution from the local equilibrium and by bouncing back of the non-equilibrium parts.