Solving the Black-Scholes equation: a demystification

François Coppex[*]
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Our objective is to show all the details of the derivation of the solution to the Black-Scholes equation without any prior prerequisit. We first show how to transform the Black-Scholes equation into a diffusion equation by means of change of variables. Applying the Fourier transform method we then find the general solution of the diffusion equation for arbitrary initial conditions. Finally, as a particular case of the general solution we establish the explicit closed-form solution for European and binary options.

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[*]URL: http://www.francoiscoppex.com

I. INTRODUCTION

While the derivation of the Black-Scholes (BS) equation can be found in many textbooks it may be harder to come across with a detailed presentation of all steps of the derivation of its solution. This article therefore aims at providing a self-contained set of explanations with all necessary details in order to establish the solution of the BS equation for vanilla options. A great deal of attention is put into chosing a route that does not require any prior knowledge of mathematical theorems or results. However a good ease with mathematics still remains a prerequisit.

The hypothesis, motivations and other considerations behind the BS equation are supposed known to the reader and will not be discussed here. Moreover we do not discuss any aspect that is not strictly related to the process of establishing the solution. Our starting point will therefore be the BS equation which is taken for granted:

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad S \geq 0, \quad t \in [0,T],} \tag{1}$$

where $V(S,t)$ is the value of the option, S the price of the underlying, t the time, T the expiration date, σ the volatility of the underlying and r the risk-free interest rate. The BS Eq. (1) is a linear parabolic equation of the form

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} + bv, \tag{2}$$

which can always be reduced to a diffusion equation

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2}. \tag{3}$$

Then an integral form of the solution of the diffusion Eq. (3) can be obtained for any initial condition using for example Fourier transform methods.

II. REDUCTION OF THE BLACK-SCHOLES EQUATION TO A DIFFUSION EQUATION

A. Reduction to a general parabolic equation

We suppose r and σ to be constant and consider the following change of variables:

$$S = Ke^x, \tag{4a}$$

$$V(S,t) = Kv(x,\tau), \tag{4b}$$

$$\tau = (T-t)\sigma^2/2. \tag{4c}$$

The partial derivatives of $V(S,t)$ therefore read

$$\frac{\partial V}{\partial t} \overset{(4b)}{=} K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} \overset{(4c)}{=} -K\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}, \tag{5a}$$

$$\frac{\partial V}{\partial S} \overset{(4b)}{=} K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} \overset{(4a)}{=} K\frac{\partial v}{\partial x}\frac{\partial}{\partial S}\ln(S/K) = \frac{K}{S}\frac{\partial v}{\partial x}, \tag{5b}$$

$$\frac{\partial^2 V}{\partial S^2} \overset{(5b)}{=} \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial v}{\partial x}\right) = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\frac{\partial}{\partial S}\frac{\partial v}{\partial x} = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S}\left(\frac{\partial x}{\partial S}\frac{\partial}{\partial x}\right)\frac{\partial v}{\partial x} = -\frac{K}{S^2}\frac{\partial v}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}. \tag{5c}$$

Inserting Eqs. (5) into Eq. (1) gives

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v. \tag{6}$$

We define

$$a = \frac{2r}{\sigma^2} - 1, \tag{7a}$$

$$b = -\frac{2r}{\sigma^2} = -(1+a), \tag{7b}$$

therefore Eq. (6) takes the form of Eq. (2):

$$\boxed{\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} + bv.} \tag{8}$$

B. Reduction to a diffusion equation

The solution of Eq. (2) must be of the form

$$\boxed{v(x,\tau) = f(\tau)g(x)h(x,\tau).} \tag{9}$$

The partial derivatives of $v(x,\tau)$ therefore read

$$\frac{\partial v}{\partial \tau} = (\partial_\tau f)gh + fg(\partial_\tau h), \tag{10a}$$

$$\frac{\partial v}{\partial x} = f(\partial_x g)h + fg(\partial_x h), \tag{10b}$$

$$\frac{\partial^2 v}{\partial x^2} = f(\partial_x^2 g)h + 2f(\partial_x g)(\partial_x h) + fg(\partial_x^2 h), \tag{10c}$$

where we have made use of the shorthand notation $\partial_x^n f = \partial^n f/\partial x^n$ and have omitted explicit functional dependence in order to lighten the notation. Inserting Eqs. (10) into Eq. (8) gives

$$(\partial_\tau f)gh + fg(\partial_\tau h) = f(\partial_x^2 g)h + 2f(\partial_x g)(\partial_x h) + fg(\partial_x^2 h) + af(\partial_x g)h + afg(\partial_x h) + bfgh. \tag{11}$$

Eq. (11) can be satisfied if f and g are of the form

$$f(\tau) = c_1 \exp[\hat{f}(\tau)], \tag{12a}$$

$$g(x) = c_2 \exp[\hat{g}(x)], \tag{12b}$$

where $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$ are two constants. Therefore:

$$\partial_\tau f(\tau) = f(\partial_\tau \hat{f}), \tag{13a}$$

$$\partial_x g(x) = g(\partial_x \hat{g}), \tag{13b}$$

$$\partial_x^2 g(x) = g(\partial_x^2 \hat{g}) + g(\partial_x \hat{g})^2. \tag{13c}$$

Inserting Eqs. (12) and (13) into Eq. (11) gives

$$(\partial_\tau h) = (\partial_x^2 h) + (\partial_x h)\big[2(\partial_g \hat{g}) + a\big] + h\big[-(\partial_\tau \hat{f}) + (\partial_x^2 \hat{g}) + (\partial_x \hat{g})^2 + a(\partial_x \hat{g}) + b\big]. \tag{14}$$

In order for Eq. (14) to be of the form of Eq. (3), it is required that:

$$2(\partial_x \hat{g}) + a = 0 \implies \hat{g}(x) = -ax/2 + c_1, \qquad c_1 \in \mathbb{R}, \tag{15a}$$

$$-(\partial_\tau \hat{f}) + (\partial_x^2 \hat{g}) + (\partial_x \hat{g})^2 + a(\partial_x \hat{g}) + b = 0 \implies \hat{f}(\tau) = (b - a^2/4)\tau + c_2, \qquad c_2 \in \mathbb{R}. \tag{15b}$$

Putting back Eqs. (15) into the solution (9) and expressing $\hat{f}(\tau)$ in terms of a only with Eq. (7b), one gets

$$v(x,\tau) = c_1 \mathrm{e}^{-(a^2/4+a+1)\tau} \mathrm{e}^{-(a/2)x} h(x,\tau), \qquad c_1 \in \mathbb{R}. \tag{16}$$

In order to sum up, the BS equation takes the form of a diffusion equation with the following change of variables:

$$\boxed{\begin{aligned} \frac{\partial h}{\partial \tau} &= \frac{\partial^2 h}{\partial x^2}, \qquad x \in \mathbb{R},\ \tau \in [0, \sigma^2 T/2], \\ S &= K\mathrm{e}^x, \\ \tau &= (T-t)\sigma^2/2, \\ V(S,t) &= K\mathrm{e}^{-(a^2/4+a+1)\tau}\mathrm{e}^{-(a/2)x} h(x,\tau), \\ a &= 2r/\sigma^2 - 1. \end{aligned}} \tag{17}$$

III. SOLUTION OF THE DIFFUSION EQUATION

A. Fourier transform of the diffusion equation

The diffusion equation (17) with initial condition $h(x,0)$ can be solved in a very straightforward way using Fourier transforms. The reader may refer to App. A for a reminder of the Fourier transform properties used in this article. If we note $\mathscr{F}(h)=\widetilde{h}(k)$ the Fourier transform of function f by respect to variable x, then thanks to the property (A3) the Fourier transform of the diffusion equation is

$$\frac{\partial \widetilde{h}}{\partial \tau} = -k^2 \widetilde{h}. \tag{18}$$

The solution of this equation is

$$\widetilde{h}(k,\tau) = \widetilde{h}(k,0)\mathrm{e}^{-k^2\tau}, \tag{19}$$

where $\widetilde{h}(k,0)$ is the Fourier transform of the initial condition for h. Translated into the original variables this initial condition corresponds to the terminal condition at expiry $t=T$ of the option, i.e. to the payoff profile at expiry as we shall see in Sects. IV and V.

B. Inverse Fourier transform of the diffusion equation

In order to find the solution for $h(x,\tau)$ it is required to apply the inverse Fourier transform of Eq. (19). If we define

$$\mathscr{F}(h_1) = \widetilde{h}_1 = \mathrm{e}^{-k^2\tau}, \tag{20a}$$

$$\mathscr{F}(h_2) = \widetilde{h}_2 = \widetilde{h}(k,0), \tag{20b}$$

then Eq. (19) can be written as

$$\widetilde{h}(k,\tau) = \widetilde{h}_1(k,\tau)\widetilde{h}_2(k,\tau). \tag{21}$$

Applying the convolution theorem as reminded in App. A 3, the inverse Fourier transform of Eq. (21) gives

$$h(x,\tau) = (h_1 * h_2)(x,\tau) \overset{(A6)}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\xi \, h_1(x-\xi,\tau) h_2(\xi,\tau). \tag{22}$$

The explicit calculation of the inverse Fourier transform of Eq. (20a) is shown in App. A 4:

$$\mathscr{F}^{-1}(\widetilde{h}_1) = h_1 = \frac{1}{\sqrt{2\tau}} \mathrm{e}^{-x^2/(4\tau)}, \tag{23a}$$

$$\mathscr{F}^{-1}(\widetilde{h}_2) = h_2 = h(x,0). \tag{23b}$$

Inserting Eqs. (23) in Eq. (22) finally gives the general solution:

$$\boxed{h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} \mathrm{d}\xi \, \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] h(\xi,0).} \tag{24}$$

As a reminder the change of variables is summarized by Eqs. (17). As shown in App. B it is easy to explicitely verify that the solution (24) satifies the diffusion Eq. (3).

IV. APPLICATION TO EUROPEAN OPTIONS

A. Initial conditions: transformation of the payoff functions

According to the change of variables (17), the initial payoff condition at $\tau=0$ corresponds to the payoff at expiry $t=T$. The payoff at expiry for European options is given by

$$V(S,T) = \max[\varepsilon(S-K),0], \tag{25}$$

where $\varepsilon = 1$ for a call option and $\varepsilon = -1$ for a put option. From Eqs. (17) we can express Eq. (25) in terms of the new variables:

$$
\begin{aligned}
h(x,0) &\overset{(17)}{=} \frac{1}{K}\mathrm{e}^{(a/2)x} V\left(K\mathrm{e}^{x}, T\right) \\
&\overset{(25)}{=} \frac{1}{K}\mathrm{e}^{(a/2)x} \max\left[\varepsilon(K\mathrm{e}^{x} - K), 0\right] \\
&= \max\left\{\varepsilon\left[\mathrm{e}^{(a/2+1)x} - \mathrm{e}^{(a/2)x}\right], 0\right\}. \qquad (26)
\end{aligned}
$$

B. Calculation of the integrals

Replacing Eq. (26) into the general solution (24) gives

$$
h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} \mathrm{d}\xi \, \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] \max\left\{\varepsilon\left[\mathrm{e}^{(a/2+1)\xi} - \mathrm{e}^{(a/2)\xi}\right], 0\right\}. \qquad (27)
$$

Since $\mathrm{e}^{(\alpha+1)x} - \mathrm{e}^{\alpha x} > 0$ if and only if $x > 0$ for all $\alpha \in \mathbb{R}$ then the integration domain of Eq. (27) can equivalently be rewritten as $[0, \infty[$. Changing variables to $\eta = \xi/\varepsilon$, $\mathrm{d}\eta = \mathrm{d}\xi/\varepsilon$ gives

$$
\begin{aligned}
h(x,\tau) &= \varepsilon\frac{1}{\sqrt{4\pi\tau}} \int_0^{\infty} \mathrm{d}\eta \, \exp\left[-\frac{(x-\varepsilon\eta)^2}{4\tau}\right] \left[\mathrm{e}^{\varepsilon(a/2+1)\eta} - \mathrm{e}^{\varepsilon(a/2)\eta}\right] \qquad (28) \\
&= \varepsilon I_{a/2+1} - \varepsilon I_{a/2}, \qquad (29)
\end{aligned}
$$

where the integrals I_α are calculated in App. D. Eq. (29) therefore becomes

$$
h(x,\tau) \overset{(D1)}{=} \varepsilon\mathrm{e}^{(a/2+1)(x+a\tau/2+\tau)} \Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right) - \varepsilon\mathrm{e}^{(a/2)(x+a\tau/2)} \Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right) \qquad (30)
$$

where Φ denotes the cumulative standard normal distribution function given by Eq. (D2).

C. Inverse change of variables

Going back to the initial variables with Eq. (17) one gets

$$
\begin{aligned}
V(S,t) &\overset{(17)}{=} \varepsilon K\mathrm{e}^{-(a^2/4+a+1)\tau}\mathrm{e}^{-(a/2)x}\left[\mathrm{e}^{(a/2+1)(x+a\tau/2+\tau)}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right) - \mathrm{e}^{(a/2)(x+a\tau/2)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\right] \\
&= \varepsilon K\mathrm{e}^{x}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right) - \varepsilon K\mathrm{e}^{-(a+1)\tau}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right) \\
&= \varepsilon K\frac{S}{K}\Phi\Bigg(\varepsilon\underbrace{\frac{\ln(S/K)+(T-t)(r-\sigma^2/2)}{\sigma\sqrt{T-t}}}_{\doteqdot d_1}\Bigg) - \varepsilon K\mathrm{e}^{-r(T-t)}\Phi\Bigg(\varepsilon\underbrace{\frac{\ln(S/K)+(T-t)(r+\sigma^2/2)}{\sigma\sqrt{T-t}}}_{\doteqdot d_2}\Bigg) \\
&= \varepsilon S\Phi(\varepsilon d_1) - \varepsilon K\mathrm{e}^{-r(T-t)}\Phi(\varepsilon d_2). \qquad (31)
\end{aligned}
$$

In order to sum-up, the solution of the Black-Scholes equation for European options is given by

$$
\boxed{
\begin{aligned}
V(S,t) &= \varepsilon S\Phi(\varepsilon d_1) - \varepsilon K\mathrm{e}^{-r(T-t)}\Phi(\varepsilon d_2), \\
d_1 &= \frac{\ln(S/K)+(T-t)(r-\sigma^2/2)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\ln(S/K)+(T-t)(r+\sigma^2/2)}{\sigma\sqrt{T-t}}, \\
\Phi(\zeta) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\zeta}\mathrm{d}\eta\,\mathrm{e}^{-\eta^2/2}, \\
\varepsilon &= \begin{cases} 1 & \text{for a call,} \\ -1 & \text{for a put.} \end{cases}
\end{aligned}
} \qquad (32)
$$

V. APPLICATION TO CASH-OR-NOTHING BINARY OPTIONS

As a further example we consider a binary option whose payoff at expiry is equal to 1 if it is in the money and 0 otherwise. The initial condition therefore reads

$$V(S,T) = \begin{cases} \Theta(S-K) & \text{for a call,} \\ 1-\Theta(S-K) & \text{for a put,} \end{cases} \tag{33}$$

where $\Theta(\zeta)$ is the Heaviside distribution which is equal to 0 if $\zeta < 0$ and is equal to 1 if $\zeta > 0$. From Eqs. (17) we can express Eq. (33) in terms of the new variables. We first note that

$$\begin{aligned} \Theta(S-K) &\overset{(17)}{=} \Theta(K\mathrm{e}^x - K) \\ &= \Theta(\mathrm{e}^x - 1) \\ &= \Theta(x), \end{aligned} \tag{34}$$

where we have made use of the properties $\Theta(\alpha\zeta) = \Theta(\zeta)$ for $\alpha \in \mathbb{R}^+$ and $\mathrm{e}^x - 1 > 0$ if and only if $x > 0$. If we further note that $1 - \Theta(x) = \Theta(-x)$, the initial condition (33) in the new variables reads

$$h(x,0) = \frac{1}{K}\mathrm{e}^{(a/2)x}\Theta(\varepsilon x), \tag{35}$$

where $\varepsilon = 1$ for a call and $\varepsilon = -1$ for a put. Replacing Eq. (35) into the general solution (24) gives

$$\begin{aligned} h(x,\tau) &= \frac{1}{K}\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \mathrm{d}\xi \, \exp\left[-\frac{(x-\xi)^2}{4\tau}\right]\mathrm{e}^{(a/2)\xi}\Theta(\varepsilon\xi) \\ &\overset{\eta=\xi/\varepsilon}{=} \frac{1}{K}\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} \mathrm{d}\eta \, \exp\left[-\frac{(x-\varepsilon\eta)^2}{4\tau}\right]\mathrm{e}^{(a/2)\varepsilon\eta}\Theta(\eta) \\ &= \underbrace{\frac{1}{K}\frac{1}{\sqrt{4\pi\tau}}\int_0^{\infty} \mathrm{d}\eta \, \exp\left[-\frac{(x-\varepsilon\eta)^2}{4\tau}\right]\mathrm{e}^{(a/2)\varepsilon\eta}}_{\overset{(D1)}{=} I_{a/2}} \\ &\overset{(D1)}{=} \frac{1}{K}\mathrm{e}^{(a/2)(x+a\tau/2)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right). \end{aligned} \tag{36}$$

Going back to the initial variables with Eq. (17) one finally gets

$$\begin{aligned} V(S,t) &= K\mathrm{e}^{-(a^2/4+a+1)\tau}\mathrm{e}^{-(a/2)x}h(x,\tau) \\ &\overset{(36)}{=} K\mathrm{e}^{-(a^2/4+a+1)\tau}\mathrm{e}^{-(a/2)x}\frac{1}{K}\mathrm{e}^{(a/2)(x+a\tau/2)}\Phi(\varepsilon d_2) \\ &= \mathrm{e}^{-r(T-t)}\Phi(\varepsilon d_2). \end{aligned} \tag{37}$$

Note that for a call (put) option with $\varepsilon = 1$ (with $\varepsilon = -1$) one gets as expected the discounted risk neutral probability that the stock price S is above (below) K at time T.

APPENDIX A: SOME PROPERTIES OF THE FOURIER TRANSFORM

1. Definition

The one-dimensional Fourier transform $\mathscr{F}(f(x))(k)$ of a function $f(x)$ such that $\int_{\mathbb{R}} \mathrm{d}x |f(x)|^2 < \infty$ is defined by

$$\mathscr{F}(f(x))(k) = \widetilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-ikx} f(x), \tag{A1}$$

where $i = \sqrt{-1}$. The inverse Fourier transform is defined by

$$\mathscr{F}^{-1}(\widetilde{f}(k))(x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k\, \mathrm{e}^{ikx} \widetilde{f}(k). \tag{A2}$$

2. The Fourier transform of derivatives

The Fourier transform of the n-th derivative of a function f is given by

$$\boxed{\mathscr{F}\left(f^{(n)}(x)\right)(k) = (ik)^n \mathscr{F}(f(x))(k).} \tag{A3}$$

This relation can be proven by successive integrations by parts. As a reminder of integration by parts, for two functions f ang g with the appropriate regularity properties we have

$$\int fg' = fg - \int f'g \tag{A4}$$

where f' denotes the derivative of f. From the definition (A1) and the regularity condition $\int_{\mathbb{R}} \mathrm{d}x |f(x)|^2 < \infty$ it follows that

$$\begin{aligned}
\mathscr{F}\left(f^{(n)}(x)\right)(k) &\overset{(A1)}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-ikx} f^{(n)}(x) \\
&\overset{(A4)}{=} \underbrace{-\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, (-ik) \mathrm{e}^{-ikx} f^{(n-1)}(x)}_{=(ik)\mathscr{F}(f^{(n-1)})} \\
&\overset{(A4)}{=} (ik)\, \underbrace{(-1)\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, (-ik) \mathrm{e}^{-ikx} f^{(n-2)}(x)}_{=(ik)\mathscr{F}(f^{(n-2)})} \\
&= \ldots \\
&= (ik)^n \mathscr{F}(f(x))(k)
\end{aligned} \tag{A5}$$

which establishes the property (A3).

3. The convolution theorem

A useful property of the Fourier transform is that the Fourier transform of the convolution product of two functions f and g is equal to the product of the Fourier transforms of f and g. If we denote by $f * g$ the convolution product of f and g:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}y\, f(x-y) g(y), \tag{A6}$$

then the Fourier transform of the convolution product is

$$\boxed{\mathscr{F}(f * g) = \mathscr{F}(f)\mathscr{F}(g).} \tag{A7}$$

The convolution theorem (A7) is easily verified from the definition (A1) of the Fourier transform. The RHS of Eq. (A7) reads

$$\begin{aligned} \mathscr{F}(f)\mathscr{F}(g) &\overset{(A1)}{=} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}u\, \mathrm{e}^{-iku} f(u) \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}v\, \mathrm{e}^{-ikv} g(v) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}u\, \mathrm{d}v\, \mathrm{e}^{-ik(u+v)} f(u)g(v). \end{aligned} \tag{A8}$$

Changing variables $t = u + v$, $s = u$, $\mathrm{d}u\mathrm{d}v = |J|\mathrm{d}s\mathrm{d}t$ where the Jacobian of the transformation is

$$|J| = \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1 \tag{A9}$$

then Eq. (A8) becomes

$$\begin{aligned} \mathscr{F}(f)\mathscr{F}(g) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}t\, \mathrm{e}^{-ikt} \underbrace{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}s\, f(t-s)g(s)}_{\overset{(A6)}{=}(f*g)(t)} \\ &\overset{(A1)}{=} \mathscr{F}(f*g) \end{aligned} \tag{A10}$$

which establishes the convolution theorem (A6).

4. A useful inverse Fourier transform

This section establishes the result

$$\boxed{\mathscr{F}^{-1}\left(\mathrm{e}^{-k^2\tau}\right) = \frac{1}{\sqrt{2\tau}}\mathrm{e}^{-x^2/(4\tau)}.} \tag{A11}$$

Proof: according to the definition of the inverse Fourier transform (A2) we have

$$\mathscr{F}^{-1}\left(\mathrm{e}^{-k^2\tau}\right) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}k\, \mathrm{e}^{ikx}\mathrm{e}^{-k^2\tau}. \tag{A12}$$

By completing the squares we require that

$$-k^2\tau + ikx = A(k+B)^2 + C, \tag{A13}$$

where A, B and C have to be determined. Equating the coefficients of the successive powers in k gives the folowing set of equations for A, B and C:

$$-\tau = A, \tag{A14a}$$
$$ix = 2AB, \tag{A14b}$$
$$0 = AB^2 + C. \tag{A14c}$$

The solution of this set of equations is

$$A = -\tau, \tag{A15a}$$
$$B = -ix/(2\tau), \tag{A15b}$$
$$C = -x^2/(4\tau). \tag{A15c}$$

Replacing Eqs. (A15) into Eq. (A12) gives

$$\mathscr{F}^{-1}\left(\mathrm{e}^{-k^2\tau}\right) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/(4\tau)} \int_{\mathbb{R}} \mathrm{d}k\, \exp\left[-\tau\left(k - i\frac{x}{2\tau}\right)^2\right]. \tag{A16}$$

We further change variables to $\xi = \sqrt{\tau}[k - ix/(2\tau)]$, $\mathrm{d}\xi = \mathrm{d}k\sqrt{\tau}$, therefore Eq. (A16) becomes

$$\begin{aligned}\mathscr{F}^{-1}\left(\mathrm{e}^{-k^2\tau}\right) &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/(4\tau)}\frac{1}{\sqrt{\tau}}\int_{\mathbb{R}}\mathrm{d}\xi\,\mathrm{e}^{-\xi^2} \\ &= \frac{1}{\sqrt{2\pi\tau}}\mathrm{e}^{-x^2/(4\tau)}\underbrace{\int_{\mathbb{R}}\mathrm{d}\xi\,\mathrm{e}^{-\xi^2}}_{=\sqrt{\pi}} \\ &= \frac{1}{\sqrt{2\tau}}\mathrm{e}^{-x^2/(4\tau)} \end{aligned} \tag{A17}$$

which establishes Eq. (A11).

Note that we have made use of the result

$$\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{-x^2} = \sqrt{\pi} \tag{A18}$$

which may be established as follows. If we note $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ then on one hand we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathbf{x}^2} &= \int_{\mathbb{R}}\mathrm{d}x_1\int_{\mathbb{R}}\mathrm{d}x_2\,\mathrm{e}^{-x_1^2-x_2^2} \\ &= \int_{\mathbb{R}}\mathrm{d}x_1\mathrm{e}^{-x_1^2}\int_{\mathbb{R}}\mathrm{d}x_2\,\mathrm{e}^{-x_2^2} \\ &= \left(\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{-x^2}\right)^2. \end{aligned} \tag{A19}$$

On the other hand, using polar coordinates $\mathbf{x} = (x_1, x_2) = r(\cos\theta, \sin\theta)$, $\mathrm{d}x_1\mathrm{d}x_2 = r\mathrm{d}r\mathrm{d}\theta$, $r \in [0, \infty[$, $\theta \in [0, 2\pi[$ we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathbf{x}^2} &= \int_0^\infty \mathrm{d}r\,r\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{-r^2} \\ &= 2\pi\int_0^\infty \mathrm{d}r\,r\mathrm{e}^{-r^2} \\ &= 2\pi\int_0^\infty \mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{1}{2}\mathrm{e}^{-r^2}\right) \\ &= \pi. \end{aligned} \tag{A20}$$

Equating the RHS of Eqs. (A19) and (A20) finally gives the result (A18).

APPENDIX B: DIRECT VERIFICATION OF THE GENERAL SOLUTION OF THE DIFFUSION EQUATION

It is easy to verify that the general solution (24) satifies the diffusion Eq. (3) with initial condition $h(x, 0)$. We have:

$$\frac{\partial h}{\partial \tau} \overset{(24)}{=} -\frac{1}{2\tau}h + \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}\mathrm{d}\xi\,\frac{(x-\xi)^2}{4\tau^2}\exp\left[-\frac{(x-\xi)^2}{4\tau}\right]h_0(\xi) = -\frac{1}{2\tau}h + \frac{1}{4\tau^2}\left\langle (x-\xi)^2\right\rangle \tag{B1a}$$

$$\frac{\partial h}{\partial x} \overset{(24)}{=} \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}\mathrm{d}\xi\,\frac{-(x-\xi)}{2\tau}\exp\left[-\frac{(x-\xi)^2}{4\tau}\right]h_0(\xi) = -\frac{1}{2\tau}\left\langle x-\xi\right\rangle, \tag{B1b}$$

$$\frac{\partial^2 h}{\partial x^2} \overset{(24)}{\underset{(B1b)}{=}} -\frac{1}{2\tau}h - \frac{1}{2\tau}\left(-\frac{1}{2\tau}\right)\left\langle (x-\xi)^2\right\rangle = -\frac{1}{2\tau}h + \frac{1}{4\tau^2}\left\langle (x-\xi)^2\right\rangle, \tag{B1c}$$

where we have made use of the shorthand notation $\langle A\rangle$ for the average of A with the measure defined by Eq. (24). It is now straightforward to see that the set of Eqs. (B1a) and (B1c) satisfies the diffusion Eq. (3).

It remains to verify the initial condition $h(x,0)$. From Eq. (24):

$$\lim_{\tau\to 0} h(x,\tau) = \int_{\mathbb{R}} \mathrm{d}\xi \underbrace{\lim_{\tau\to 0} \frac{1}{\sqrt{4\pi\tau}} \left\{ \exp\left[-\frac{(x-\xi)^2}{4\tau} \right] \right\}}_{=\delta(\xi-x)} h(\xi,0) \tag{B2}$$

$$= h(x,0), \tag{B3}$$

where $\delta(\xi - x)$ is the Dirac distribution. The reader may refer to Appendix C for a proof of property (B2).

APPENDIX C: A USEFUL PROPERTY OF THE DIRAC DISTRIBUTION

The Dirac distribution $\delta(x)$ (or "delta function") is defined by

$$\int_{\mathbb{R}} \mathrm{d}x\, \varphi(x)\delta(x-x_0) = \varphi(x_0) \tag{C1}$$

for each continuous function $\varphi(x)$. A Dirac distribution may be obtained from the following process: for all $f(x)$ such that

$$\int_{\mathbb{R}} \mathrm{d}x\, f(x) = 1 \tag{C2}$$

then

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} f\left(\frac{x-x_0}{\varepsilon}\right) = \delta(x-x_0). \tag{C3}$$

We made use of this property in Eq. (B3) with $\varepsilon = \tau^{1/2}$. In order to establish property (C3) we first consider the change of variables $y = (x-x_0)/\varepsilon$, $\mathrm{d}y = \mathrm{d}x/\varepsilon$, therefore

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \mathrm{d}x\, \varphi(x) \frac{1}{\varepsilon} f\left(\frac{x-x_0}{\varepsilon}\right) &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \mathrm{d}y\, \varphi(y\varepsilon + x_0) f(y) \\
&= \int_{\mathbb{R}} \mathrm{d}y\, \lim_{\varepsilon\to 0} \varphi(y\varepsilon + x_0) f(y) \\
&= \int_{\mathbb{R}} \mathrm{d}y\, \varphi(x_0) f(y) \\
&= \varphi(x_0) \underbrace{\int_{\mathbb{R}} \mathrm{d}y\, f(y)}_{\overset{(C2)}{=} 1} \\
&= \varphi(x_0)
\end{aligned} \tag{C4}$$

where we are allowed by the dominated convergence theorem to intervene the limit and integration. This establishes Eq. (C3).

APPENDIX D: CALCULATION OF A CLASS OF GAUSSIAN INTEGRALS

We establish the following result

$$\boxed{I_\alpha = \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty \mathrm{d}\xi\, \exp\left[-\frac{(x-\varepsilon\xi)^2}{4\tau} + \alpha\varepsilon\xi \right] = \mathrm{e}^{\alpha(x+\alpha\tau)} \Phi\left(\varepsilon \frac{x+2\tau\alpha}{\sqrt{2\tau}} \right),} \tag{D1}$$

where $\varepsilon = \pm 1$ and Φ denotes the cumulative standard normal distribution function

$$\Phi(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} \mathrm{d}\eta\, \mathrm{e}^{-\eta^2/2}. \tag{D2}$$

Proof: by completing the squares of the exponential, we require that:

$$-\frac{(x-\varepsilon\xi)^2}{4\tau}+\alpha\varepsilon\xi=-c_1(c_2-\varepsilon\xi)^2+c_3, \tag{D3}$$

where c_i, $i=1,\dots,3$ are constants by respect to the integration variable. Expanding Eq. (D3) gives

$$-\frac{1}{4\tau}(\varepsilon\xi)^2+(\varepsilon\xi)\left(\alpha+\frac{x}{2\tau}\right)-\frac{1}{4\tau}x^2=-c_1\xi^2+\xi 2c_1c_2-c_1c_2^2+c_3. \tag{D4}$$

Equating powers in $\varepsilon\xi$ in Eq. (D4) gives

$$c_1 = 1/(4\tau), \tag{D5a}$$
$$c_2 = x+2\tau\alpha, \tag{D5b}$$
$$c_3 = \alpha(x+\alpha\tau). \tag{D5c}$$

Eq. (D1) therefore now reads

$$I_\alpha=\mathrm{e}^{c_3}\frac{1}{\sqrt{4\pi\tau}}\int_0^\infty \mathrm{d}\xi\,\mathrm{e}^{-c_1(c_2-\varepsilon\xi)^2}. \tag{D6}$$

Changing variables to $\eta=(c_2-\varepsilon\xi)\sqrt{2c_1}$, $\mathrm{d}\eta=-\mathrm{d}\xi\varepsilon\sqrt{2c_1}$, Eq. (D6) becomes

$$I_\alpha=\frac{\mathrm{e}^{c_3}}{\sqrt{2c_1}}\frac{1}{\sqrt{4\pi\tau}}\left(-\varepsilon\int_{c_2\sqrt{2c_1}}^{-\varepsilon\infty}\mathrm{d}\eta\,\mathrm{e}^{-\eta^2/2}\right). \tag{D7}$$

If $\varepsilon=1$ Eq. (D7) becomes

$$\begin{aligned} I_\alpha &= \mathrm{e}^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c_2\sqrt{2c_1}}\mathrm{d}\eta\,\mathrm{e}^{-\eta^2/2} \\ &= \mathrm{e}^{c_3}\Phi\left(c_2\sqrt{2c_1}\right). \end{aligned} \tag{D8}$$

If $\varepsilon=-1$ Eq. (D7) becomes

$$I_\alpha=\mathrm{e}^{c_3}\frac{1}{\sqrt{2\pi}}\int_{c_2\sqrt{2c_1}}^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-\eta^2/2}. \tag{D9}$$

Using the change of variables $\zeta=-\eta$, $\mathrm{d}\zeta=-\mathrm{d}\eta$, Eq. (D9) becomes

$$\begin{aligned} I_\alpha &= \mathrm{e}^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-c_2\sqrt{2c_1}}\mathrm{d}\zeta\,\mathrm{e}^{-\zeta^2/2} \\ &= \mathrm{e}^{c_3}\Phi\left(-c_2\sqrt{2c_1}\right). \end{aligned} \tag{D10}$$

Combining Eqs. (D8) and (D10) gives

$$I_\alpha=\mathrm{e}^{c_3}\Phi\left(\varepsilon c_2\sqrt{2c_1}\right), \tag{D11}$$

where

$$\mathrm{e}^{c_3} \overset{(D5)}{=} \mathrm{e}^{\alpha(x+\alpha\tau)}, \tag{D12a}$$
$$c_2\sqrt{2c_1} \overset{(D5)}{=} \frac{x+2\tau\alpha}{\sqrt{2\tau}}. \tag{D12b}$$

Replacing Eqs. (D12) in Eq. (D11) gives the result (D1).